Комбінаторика

1)

For any positive integer k, A_k is the subset of $\{k+1, k+2, \ldots, 2k\}$ consisting of all elements whose digits in base 2 contain exactly three 1's. Let $f(k)$ denote the number of elements in A_k.

(a) Prove that for any positive integer m, $f(k) = m$ has at least one solution.

(b) Determine all positive integers m for which $f(k) = m$ has a unique solution.

2)

Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts. Can Peter always transfer all his money into one account?

3)

1994 girls are seated at a round table. Initially one girl holds n tokens. Each turn a girl who is holding more than one token passes one token to each of her neighbours.

a.) Show that if $n < 1994$, the game must terminate. b.) Show that if $n = 1994$ it cannot terminate.

4)

Let p be an odd prime. Find the number of p-element subsets A of $\{1, 2, \ldots, 2p\}$ such that the sum of all elements of A is divisible by p.

5)

We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20$, $|BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: One can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square corresponding to vertex A to the square corresponding to vertex B.

(a) Show that the task cannot be done if r is divisible by 2 or 3.

(b) Prove that the task is possible when $r = 73$.

(c) Is there a solution when $r = 97$?

6)

An $n \times n$ matrix with entries from $\{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if for each i the union of the ith row and the ith column contains $2n-1$ distinct entries. Show that:

(a) There exist no silver matrices for $n = 1997$.

(b) Silver matrices exist for infinitely many values of n.

7)

For a positive integer n, let $f(n)$ denote the number of ways to represent n as the sum of powers of 2 with nonnegative integer exponents. Representations that differ only in the ordering in their summands are not considered to be distinct. (For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.) Prove that the inequality

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}$$

holds for any integer $n \geq 3$.

8)

In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

9)

C3 (AUS) The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices (where each pair of vertices are joined by an edge).

10)

A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

11)

Consider a $m \times n$ rectangular board consisting of mn unit squares. Two of its unit squares are called *adjacent* if they have a common edge, and a *path* is a sequence of unit squares in which any two consecutive squares are adjacent. Two parths are called *non-intersecting* if they don't share any common squares.

Each unit square of the rectangular board can be colored black or white. We speak of a *coloring* of the board if all its mn unit squares are colored.

Let N be the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge. Let M be the number of colorings of the board for which there exist at least two non-intersecting black paths from the left edge of the board to its right edge.

Prove that $N^2 \geq M \cdot 2^{mn}$.

12)

In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.