Inequalities. IMO preparation

1. (All-Russian 2004) Let $n > 3$ be a natural number and let $x_1, ..., x_n$ be positive numbers with product equal 1. Prove that

$$\frac{1}{1+x_1+x_1 \cdot x_2} + \frac{1}{1+x_2+x_2 \cdot x_3} + ... + \frac{1}{1+x_n+x_n \cdot x_1} > 1.$$

2. Let $r_1, ...r_n$ be nonnegative numbers. Then for arbitrary real numbers $x_1, ..., x_n$ the following inequality holds

$$\sum_{i,j} \min(r_i, r_j) x_i x_j \geq 0.$$

3. (Iran TST 1996) For positive x, y, z prove the inequality

$$(xy + yz + zx) \left(\sum \frac{1}{(x+y)^2} \right) \geq \frac{9}{4}$$

4. (Kolmogorov Cup 2011) Let $x_1, ..., x_n$ be positive numbers with product equal 1. Prove that there exists a positive integer $k \leq n$ such that

$$\frac{x_k}{k + x_1 + ... + x_k} \geq 1 - \frac{1}{\sqrt[n]{2}}$$

5. (All-Russian 2016) For positive real numbers a, b, c, d with $a+b+c+d = 3$ prove the inequality

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \leq \frac{1}{a^3 b^3 c^3 d^3}$$

6. (FYM 2011)Let $k \leq n$ be two positive integers and $x_1, x_2, ..., x_{n+k}$ be a sequence of positive real numbers with $x_1 + x_2 + ... + x_{n+k} = n + k$. Prove that

$$\sum_{i=1}^{n} \frac{x_i^n}{x_i x_{i+1} ... x_{i+k-1} + x_{i+k} ... x_{i+n+k-1}} \geq \frac{n+k}{2}$$

(we consider cyclic enumeration of variables)

7. (China 2015) Let $x_1, x_2, \cdots, x_n$ be a non-decreasing monotonous sequence of positive numbers such that $x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}$ is a non-increasing monotonous sequence. Prove that

$$\frac{\sum_{i=1}^{n} x_i}{n \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}} \leq \frac{n+1}{2\sqrt[n]{n!}}$$

8. (IMO 2003) Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

9. (IMO 2006) Determine the least real number M such that the inequality

$$|ab(a^2-b^2) + bc(b^2-c^2) + ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$$

holds for all real numbers a,b and c.

10. (IMO 2008) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers x, y, z different from 1 and satisfying $xyz = 1$
Prove that the equality holds for infinetely many triplets of rational numbers x, y, z different from 1 and satisfying $xyz = 1$

11. (USA TST 2009) Prove that for positive real numbers x, y, z the following inequality holds:

$$x^3(y^2+z^2)^2+y^3(z^2+x^2)^2+z^3(x^2+y^2)^2 \geq xyz\left[xy(x+y)^2 + yz(y+z)^2 + zx(z+x)^2\right].$$

12. (ISL 2007) Let n ne a positive integer and let x, y be positive real numbers such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right) \cdot \left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)\cdot(1-y)}.$$

13. (USAMO 2000) Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \leq \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}.$$

14. (ISL 2007) Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \ldots + a_{100}^2 = 1$. Prove that

$$a_1^2 \cdot a_2 + a_2^2 \cdot a_3 + \ldots + a_{100}^2 \cdot a_1 < \frac{12}{25}.$$

15. (Romania TST 2014, Hardy's inequality) Let $x_1, ..., x_n$ be nonnegative real numbers. Prove that

$$x_1^2 + \left(\frac{x_1+x_2}{2}\right)^2 + ... + \left(\frac{x_1+x_2+...+x_n}{n}\right)^2 \leq 4\left(x_1^2+x_2^2+...+x_n^2\right)$$