RMM2016?

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1. Every cell of a $m \times n$ chess board, $m \geq 2, n \geq 2$, is colored with one of four possible colors, e.g white, green, red, blue. We call such coloring good if the four cells of any 2×2 square of the chessboard are colored with pairwise different colors. Determine the number of all good colorings of the chess board.

2. Let $ABCD$ be a quadrilateral inscribed in a circle k. AC and BD meet at E. The rays $\overrightarrow{CB}, \overrightarrow{DA}$ meet at F. Prove that the line through the incenters of $\triangle ABE$, $\triangle ABF$ and the line through the incenters of $\triangle CDE$, $\triangle CDF$ meet at a point lying on the circle k.

3. Prove that the natural numbers can be coloured using exactly two colours in a way that both conditions are fulfilled: 1) For every prime number p and every natural number n, the numbers p^n, p^{n+1} and p^{n+2} do not have the same colour. 2) There does not exist an infinite geometric sequence of natural numbers of the same colour.

4. Let $f_1(x)$ be a polynomial of degree 2 with the leading coefficient positive and $f_{n+1}(x) = f_1(f_n(x))$ for $n \geq 1$. Prove that if the equation $f_2(x) = 0$ has four different non-positive real roots, then for arbitrary n then $f_n(x)$ has 2^n different real roots.

5. Find all functions f from the real numbers to the real numbers which satisfy
$$f(x^3) + f(y^3) = (x + y)(f(x^2) + f(y^2) - f(xy))$$
for all real numbers x and y.

6. The integer x is at least 3 and $n = x^6 - 1$. Let p be a prime and k be a positive integer such that p^k is a factor of n. Show that $p^{3k} < 8n$.

7. Let ABC be a triangle and P be a point in its interior. Let AP meet the circumcircle of ABC again at A'. The points B' and C' are similarly defined. Let OA be the circumcentre of BCP. The circumcentres O_B and O_C are similarly defined. Let O'_A be the circumcentre of $B'C'P$. The circumcentres O'_B and O'_C are similarly defined. Prove that the lines $O_AO'_A, O_BO'_B$ and $O_CO'_C$ are concurrent.

8. The edges of a complete graph with $2n$ vertices ($n \geq 4$) are colored in blue and red such that there is no blue triangle and there is no red complete subgraph with n vertices. Find the least possible number of blue edges.