The 5^{th} Romanian Master of Mathematics Competition

Solutions of the problems

Problem 1 = 4′. Prove that there are infinitely many positive integer numbers n such that $2^{2^n+1}+1$ be divisible by n, but 2^n+1 be not.

Solution 1. Throughout the solution n stands for a positive integer. By Euler's theorem, $(2^{3^n}+1)(2^{3^n}-1) = 2^{2\cdot 3^n}-1 \equiv 0 \pmod{3^{n+1}}$. Since $2^{3^n}-1 \equiv 1 \pmod 3$, it follows that $2^{3^n}+1$ is divisible by 3^{n+1}.

The number $(2^{3^{n+1}}+1)/(2^{3^n}+1) = 2^{2\cdot 3^n} - 2^{3^n} + 1$ is greater than 3 and congruent to 3 modulo 9, so it has a prime factor $p_n > 3$ that does not divide $2^{3^n}+1$ (otherwise, $2^{3^n} \equiv -1 \pmod{p_n}$, so $2^{2\cdot 3^n} - 2^{3^n} + 1 \equiv 3 \pmod{p_n}$, contradicting the fact that p_n is a factor greater than 3 of $2^{2\cdot 3^n} - 2^{3^n} + 1$).

We now show that $a_n = 3^n p_n$ satisfies the conditions in the statement. Since $2^{a_n}+1 \equiv 2^{3^n}+1 \not\equiv 0 \pmod{p_n}$, it follows that a_n does not divide $2^{a_n}+1$.

On the other hand, 3^{n+1} divides $2^{3^n}+1$ which in turn divides $2^{a_n}+1$, so $2^{3^{n+1}}+1$ divides $2^{2^{a_n}+1}+1$. Finally, both 3^n and p_n divide $2^{3^{n+1}}+1$, so a_n divides $2^{2^{a_n}+1}+1$.

As n runs through the positive integers, the a_n are clearly pairwise distinct and the conclusion follows.

Solution 2. (Géza Kós) We show that the numbers $a_n = (2^{3^n}+1)/9$, $n \geq 2$, satisfy the conditions in the statement. To this end, recall the following well-known facts:

(1) If N is an odd positive integer, then $\nu_3(2^N+1) = \nu_3(N)+1$, where $\nu_3(a)$ is the exponent of 3 in the decomposition of the integer a into prime factors; and

(2) If M and N are odd positive integers, then $(2^M+1, 2^N+1) = 2^{(M,N)}+1$, where (a,b) is the greatest common divisor of the integers a and b.

By (1), $a_n = 3^{n-1}m$, where m is an odd positive integer not divisible by 3, and by (2),

$$(m, 2^{a_n}+1) \mid (2^{3^n}+1, 2^{a_n}+1) = 2^{(3^n, a_n)}+1 = 2^{3^{n-1}}+1 < \frac{2^{3^n}+1}{3^{n+1}} = m,$$

so m cannot divide $2^{a_n}+1$.

On the other hand, $3^{n-1} \mid 2^{2^{a_n}+1}+1$, for $\nu_3(2^{2^{a_n}+1}+1) > \nu_3(2^{a_n}+1) > \nu_3(a_n) = n-1$, and $m \mid 2^{2^{a_n}+1}+1$, for $3^{n-1} \mid a_n$, so $3^n \mid 2^{a_n}+1$ whence $m \mid 2^{3^n}+1 \mid 2^{2^{a_n}+1}+1$. Since 3^{n-1} and m are coprime, the conclusion follows.

Remarks. There are several variations of these solutions. For instance, let $b_1 = 3$ and $b_{n+1} = 2^{b_n}+1$, $n \geq 1$, and notice that b_n divides b_{n+1}. It can be shown that there are infinitely many indices n such that some prime factor p_n of b_{n+1} does not divide b_n. One checks that for these n's the $a_n = p_n b_{n-1}$ satisfy the required conditions.

Finally, the numbers $3^n \cdot 571$, $n \geq 2$, form yet another infinite set of positive integers fulfilling the conditions in the statement — the details are omitted.

Problem 2 = 5′. Given a positive integer number $n \geq 3$, colour each cell of an $n \times n$ square array one of $[(n+2)^2/3]$ colours, each colour being used at least once. Prove that the cells of some 1×3 or 3×1 rectangular subarray have pairwise distinct colours.

Solution. For more convenience, say that a subarray of the $n \times n$ square array *bears* a colour if at least two of its cells share that colour.

We shall prove that the number of 1×3 and 3×1 rectangular subarrays, which is $2n(n-2)$, exceeds the number of such subarrays, each of which bears some colour. The key ingredient is the estimate in the lemma below.

Lemma. *If a colour is used exactly p times, then the number of 1×3 and 3×1 rectangular subarrays bearing that colour does not exceed $3(p-1)$.*

Assume the lemma for the moment, let $N = [(n+2)^2/3]$ and let n_i be the number of cells coloured the ith colour, $i = 1, \ldots, N$, to deduce that the number of 1×3 and 3×1 rectangular subarrays, each of which bears some colour, is at most

$$\sum_{i=1}^{N} 3(n_i - 1) = 3\sum_{i=1}^{N} n_i - 3N = 3n^2 - 3N < 3n^2 - (n^2 + 4n) = 2n(n-2)$$

and thereby conclude the proof.

Back to the lemma, the assertion is clear if $p = 1$, so let $p > 1$.

We begin by showing that if a row contains exactly q cells coloured C, then the number r of 3×1 rectangular subarrays bearing C does not exceed $3q/2 - 1$; of course, a similar estimate holds for a column. To this end, notice first that the case $q = 1$ is trivial, so we assume that $q > 1$. Consider the incidence of a cell c coloured C and a 3×1 rectangular subarray R bearing C:

$$\langle c, R \rangle = \begin{cases} 1 & \text{if } c \subset R, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, given R, $\sum_c \langle c, R \rangle \geq 2$, and, given c, $\sum_R \langle c, R \rangle \leq 3$; moreover, if c is the leftmost or rightmost cell, then $\sum_R \langle c, R \rangle \leq 2$. Consequently,

$$2r \leq \sum_R \sum_c \langle c, R \rangle = \sum_c \sum_R \langle c, R \rangle \leq 2 + 3(q-2) + 2 = 3q - 2,$$

whence the conclusion.

Finally, let the p cells coloured C lie on k rows and ℓ columns and notice that $k + \ell \geq 3$, for $p > 1$. By the preceding, the total number of 3×1 rectangular subarrays bearing C does not exceed $3p/2 - k$, and the total number of 1×3 rectangular subarrays bearing C does not exceed $3p/2 - \ell$, so the total number of 1×3 and 3×1 rectangular subarrays bearing C does not exceed $(3p/2 - k) + (3p/2 - \ell) = 3p - (k + \ell) \leq 3p - 3 = 3(p-1)$. This completes the proof.

Remarks. In terms of the total number of cells, the number $N = [(n+2)^2/3]$ of colours is asymptotically close to the minimum number of colours required for some 1×3 or 3×1 rectangular subarray to have all cells of pairwise distinct colours, whatever the colouring. To see this, colour the cells with the coordinates (i, j), where $i + j \equiv 0 \pmod 3$ and $i, j \in \{0, 1, \ldots, n-1\}$, one colour each, and use one additional colour C to colour the remaining cells. Then each 1×3 and each 3×1 rectangular subarray has exactly two cells coloured C, and the number of colours is $\lceil n^2/3 \rceil + 1$ if $n \equiv 1$ or $2 \pmod 3$, and $\lceil n^2/3 \rceil$ if $n \equiv 0 \pmod 3$. Consequently, the minimum number of colours is $n^2/3 + O(n)$.

Problem 3 = 3′. Each positive integer number is coloured red or blue. A function f from the set of positive integer numbers into itself has the following two properties:

(a) if $x \le y$, then $f(x) \le f(y)$; and

(b) if x, y and z are all (not necessarily distinct) positive integer numbers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.

Prove that there exists a positive number a such that $f(x) \le ax$ for all positive integer numbers x.

Solution. For integer x, y, by a segment $[x, y]$ we always mean the set of all integers t such that $x \le t \le y$; the *length* of this segment is $y - x$.

If for every two positive integers x, y sharing the same colour we have $f(x)/x = f(y)/y$, then one can choose $a = \max\{f(r)/r, f(b)/b\}$, where r and b are arbitrary red and blue numbers, respectively. So we can assume that there are two red numbers x, y such that $f(x)/x \ne f(y)/y$.

Set $m = xy$. Then each segment of length m contains a blue number. Indeed, assume that all the numbers on the segment $[k, k + m]$ are red. Then

$$f(k+m) = f(k+xy) = f(k+x(y-1)) + f(x) = \cdots = f(k) + yf(x),$$
$$f(k+m) = f(k+xy) = f(k+(x-1)y) + f(y) = \cdots = f(k) + xf(y),$$

so $yf(x) = xf(y)$ — a contradiction. Now we consider two cases.

Case 1. Assume that there exists a segment $[k, k + m]$ of length m consisting of blue numbers. Define $D = \max\{f(k), \ldots, f(k + m)\}$. We claim that $f(z) - f(z - 1) \le D$, whatever $z > k$, and the conclusion follows. Consider the largest blue number b_1 not exceeding z, so $z - b_1 \le m$, and some blue number b_2 on the segment $[b_1 + k, b_1 + k + m]$, so $b_2 > z$. Write $f(b_2) = f(b_1) + f(b_2 - b_1) \le f(b_1) + D$ to deduce that $f(z + 1) - f(z) \le f(b_2) - f(b_1) \le D$, as claimed.

Case 2. Each segment of length m contains numbers of both colours. Fix any red number $R \ge 2m$ such that $R + 1$ is blue and set $D = \max\{f(R), f(R + 1)\}$. Now we claim that $f(z + 1) - f(z) \le D$, whatever $z > 2m$. Consider the largest red number r not exceeding z and the largest blue number b smaller than r; then $0 < z - b = (z - r) + (r - b) \le 2m$, and $b + 1$ is red. Let $t = b + R + 1$; then $t > z$. If t is blue, then $f(t) = f(b) + f(R + 1) \le f(b) + D$, and $f(z + 1) - f(z) \le f(t) - f(b) \le D$. Otherwise, $f(t) = f(b + 1) + f(R) \le f(b + 1) + D$, hence $f(z + 1) - f(z) \le f(t) - f(b + 1) \le D$, as claimed.

Problem 4 = 1′. Given a finite group of boys and girls, a *covering set of boys* is a set of boys such that every girl knows at least one boy in that set; and a *covering set of girls* is a set of girls such that every boy knows at least one girl in that set. Prove that the number of covering sets of boys and the number of covering sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

Solution 1. A set X of boys is *separated* from a set Y of girls if no boy in X is an acquaintance of a girl in Y. Similarly, a set Y of girls is *separated* from a set X of boys if no girl in Y is an acquaintance of a boy in X. Since acquaintance is assumed mutual, separation is symmetric: X is separated from Y if and only if Y is separated from X.

This enables doubly counting the number n of ordered pairs (X, Y) of separated sets X, of boys, and Y, of girls, and thereby showing that it is congruent modulo 2 to both numbers in question.

Given a set X of boys, let Y_X be the largest set of girls separated from X, to deduce that X is separated from exactly $2^{|Y_X|}$ sets of girls. Consequently, $n = \sum_X 2^{|Y_X|}$ which is clearly congruent modulo 2 to the number of covering sets of boys.

Mutatis mutandis, the argument applies to show n congruent modulo 2 to the number of covering sets of girls.

Remark. The argument in this solution translates verbatim in terms of the adjancency matrix of the associated acquaintance graph.

Solution 2. (Ilya Bogdanov) Let B denote the set of boys, let G denote the set of girls and induct on $|B| + |G|$. The assertion is vacuously true if either set is empty.

Next, fix a boy b, let $B' = B \setminus \{b\}$, and let G' be the set of all girls who do not know b. Notice that:

(1) a covering set of boys in $B' \cup G$ is still one in $B \cup G$; and

(2) a covering set of boys in $B \cup G$ which is no longer one in $B' \cup G$ is precisely the union of a covering set of boys in $B' \cup G'$ and $\{b\}$,

so the number of covering sets of boys in $B \cup G$ is the sum of those in $B' \cup G$ and $B' \cup G'$.

On the other hand,

(1′) a covering set of girls in $B \cup G$ is still one in $B' \cup G$; and

(2′) a covering set of girls in $B' \cup G$ which is no longer one in $B \cup G$ is precisely a covering set of girls in $B' \cup G'$,

so the number of covering sets of girls in $B \cup G$ is the difference of those in $B' \cup G$ and $B' \cup G'$.

Since the assertion is true for both $B' \cup G$ and $B' \cup G'$ by the induction hypothesis, the conclusion follows.

Solution 3. (Géza Kós) Let B and G denote the sets of boys and girls, respectively. For every pair $(b, g) \in B \times G$, write $f(b, g) = 0$ if they know each other, and $f(b, g) = 1$ otherwise. A set X of boys is covering if and only if

$$\prod_{g \in G} \left(1 - \prod_{b \in X} f(b, g)\right) = 1.$$

Hence the number of covering sets of boys is

$$\sum_{X\subseteq B}\prod_{g\in G}\left(1-\prod_{b\in X}f(b,g)\right)\equiv\sum_{X\subseteq B}\prod_{g\in G}\left(1+\prod_{b\in X}f(b,g)\right)$$
$$=\sum_{X\subseteq B}\sum_{Y\subseteq G}\prod_{b\in X}\prod_{g\in Y}f(b,g)\pmod 2.$$

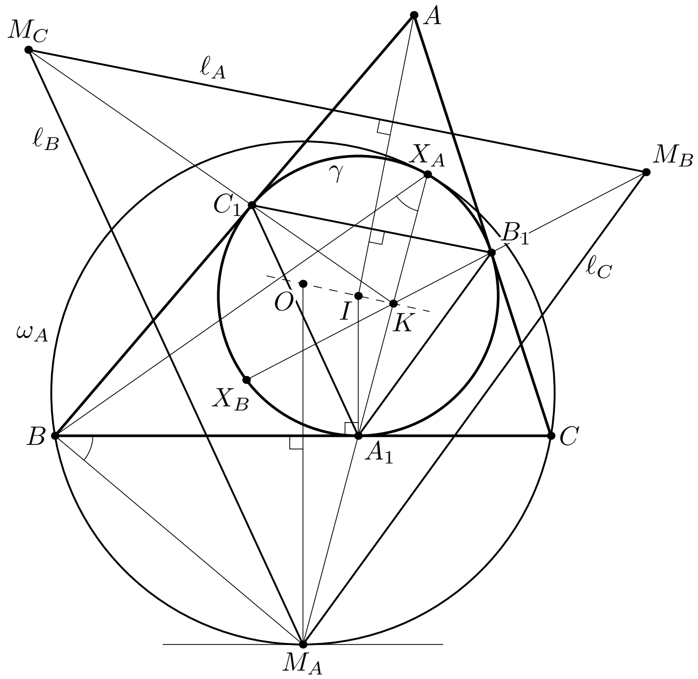
By symmetry, the same is valid for the number of covering sets of girls.

Problem 6 = 6′. Let ABC be a triangle and let I and O respectively denote its incentre and circumcentre. Let ω_A be the circle through B and C and tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C through A meet again at A'; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

Solution. Let γ be the incircle of the triangle ABC and let A_1, B_1, C_1 be its contact points with the sides BC, CA, AB, respectively. Let further X_A be the point of contact of the circles γ and ω_A. The latter circle is the image of the former under a homothety centred at X_A. This homothety sends A_1 to a point M_A on ω_A such that the tangent to ω_A at M_A is parallel to BC. Consequently, M_A is the midpoint of the arc BC of ω_A not containing X_A. It follows that the angles M_AX_AB and M_ABC are congruent, so the triangles M_ABA_1 and M_AX_AB are similar: $M_AB/M_AX_A = M_AA_1/M_AB$. Rewrite the latter $M_AB^2 = M_AA_1 \cdot M_AX_A$ to deduce that M_A lies on the radical axis ℓ_B of B and γ. Similarly, M_A lies on the radical axis ℓ_C of C and γ.

Define the points X_B, X_C, M_B, M_C and the line ℓ_A in a similar way and notice that the lines ℓ_A, ℓ_B, ℓ_C support the sides of the triangle $M_AM_BM_C$. The lines ℓ_A and B_1C_1 are both perpendicular to AI, so they are parallel. Similarly, the lines ℓ_B and ℓ_C are parallel to C_1A_1 and A_1B_1, respectively. Consequently, the triangle $M_AM_BM_C$ is the image of the triangle $A_1B_1C_1$ under a homothety Θ. Let K be the centre of Θ and let $k = M_AK/A_1K = M_BK/B_1K = M_CK/C_1K$ be the similitude ratio. Notice that the lines M_AA_1, M_BB_1 and M_CC_1 are concurrent at K.

Since the points A_1, B_1, X_A, X_B are concyclic, $A_1K \cdot KX_A = B_1K \cdot KX_B$. Multiply both sides by k to get $M_AK \cdot KX_A = M_BK \cdot KX_B$ and deduce thereby that K lies on the radical axis CC' of ω_A and ω_B. Similarly, both lines AA' and BB' pass through K.

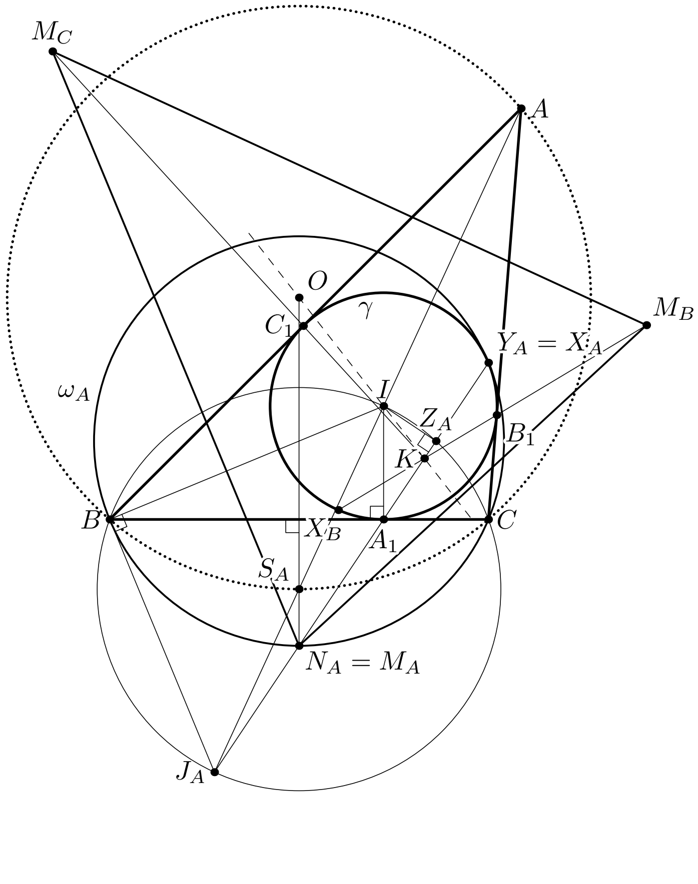


Finally, consider the image O' of I under Θ. It lies on the line through M_A parallel to A_1I (and hence perpendicular to BC); since M_A is the midpoint of the arc BC, this line must be M_AO. Similarly, O' lies on the line M_BO, so $O' = O$. Consequently, the points I, K and O are collinear.

Remark 1. Many steps in this solution allow different reasonings. For instance, one may see that the lines A_1X_A and B_1X_B are concurrent at point K on the radical axis CC' of the circles ω_A and ω_B by applying Newton's theorem to the quadrilateral $X_AX_BA_1B_1$ (since the common tangents at X_A and X_B intersect on CC'). Then one can conclude that $KA_1/KB_1 = KM_A/KM_B$, thus obtaining that the triangles $M_AM_BM_C$ and $A_1B_1C_1$ are homothetical at K (and therefore K is the radical center of ω_A, ω_B, and ω_C). Finally, considering the inversion with the pole K and the power equal to $KX_1 \cdot KM_A$ followed by the reflection at P we see that the circles ω_A, ω_B, and ω_C are invariant under this transform; next, the image of γ is the circumcircle of $M_AM_BM_C$ and it is tangent to all the circles ω_A, ω_B, and ω_C, hence its center is O, and thus O, I, and K are collinear.

Remark 2. Here is an outline of an alternative approach to the first part of the solution. Let J_A be the excentre of the triangle ABC opposite A. The line J_AA_1 meets γ again at Y_A; let Z_A and N_A be the midpoints of the segments A_1Y_A and J_AA_1, respectively. Since the segment IJ_A is a diameter in the circle BCZ_A, it follows that $BA_1 \cdot CA_1 = Z_AA_1 \cdot J_AA_1$, so $BA_1 \cdot CA_1 = N_AA_1 \cdot Y_AA_1$. Consequently, the points B, C, N_A and Y_A lie on some circle ω'_A.

It is well known that N_A lies on the perpendicular bisector of the segment BC, so the tangents to ω'_A and γ at N_A and A_1 are parallel. It follows that the tangents to these circles at Y_A coincide, so ω'_A is in fact ω_A, whence $X_A = Y_A$ and $M_A = N_A$. It is also well known that the midpoint S_A of the segment IJ_A lies both on the circumcircle ABC and on the perpendicular bisector of BC. Since S_AM_A is a midline in the triangle A_1IJ_A, it follows that $S_AM_A = r/2$, where r is the radius of γ (the inradius of the triangle ABC). Consequently, each of the points M_A, M_B and M_C is at distance $R + r/2$ from O (here R is the circumradius). Now proceed as above.



Problem 2′. Given a triangle ABC, let D, E, and F respectively denote the midpoints of the sides BC, CA, and AB. The circle BCF and the line BE meet again at P, and the circle ABE and the line AD meet again at Q. Finally, the lines DP and FQ meet at R. Prove that the centroid G of the triangle ABC lies on the circle PQR.

Solution 1. We will use the following lemma.

Lemma. *Let AD be a median in triangle ABC. Then $\cot\angle BAD = 2\cot A + \cot B$ and $\cot\angle ADC = \frac{1}{2}(\cot B - \cot C)$.*

Proof. Let CC_1 and DD_1 be the perpendiculars from C and D to AB. Using the signed lengths we write

$$\cot BAD = \frac{AD_1}{DD_1} = \frac{(AC_1 + AB)/2}{CC_1/2} = \frac{CC_1\cot A + CC_1(\cot A + \cot B)}{CC_1} = 2\cot A + \cot B.$$

Similarly, denoting by A_1 the projection of A onto BC, we get

$$\cot ADC = \frac{DA_1}{AA_1} = \frac{BC/2 - A_1C}{AA_1} = \frac{(AA_1\cot B + AA_1\cot C)/2 - AA_1\cot C}{AA_1} = \frac{\cot B - \cot C}{2}.$$

The Lemma is proved.

Turning to the solution, by the Lemma we get

$$\begin{aligned}\cot\angle BPD &= 2\cot\angle BPC + \cot\angle PBC = 2\cot\angle BFC + \cot\angle PBC \quad \text{(from circle } BFPC\text{)}\\ &= 2\cdot\frac{1}{2}(\cot A - \cot B) + 2\cot B + \cot C = \cot A + \cot B + \cot C.\end{aligned}$$

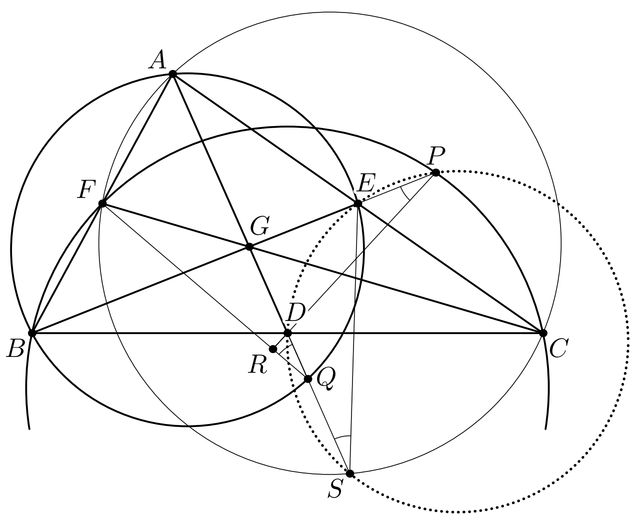
Similarly, $\cot\angle GQF = \cot A + \cot B + \cot C$, so $\angle GPR = \angle GQF$ and $GPRQ$ is cyclic.

Remark. The angle $\angle GPR = \angle GQF$ is the Brocard angle.

Solution 2. (Ilya Bogdanov and Marian Andronache) We also prove that $\angle(RP, PG) = \angle(RQ, QG)$, or $\angle(DP, PG) = \angle(FQ, QG)$.

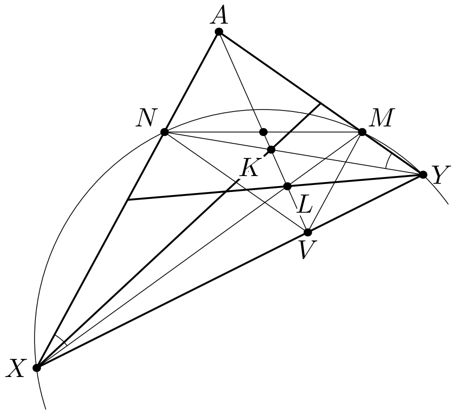
Let S be the point on ray GD such that $AG\cdot GS = CG\cdot GF$ (so the points A, S, C, F are concyclic). Then $GP\cdot GE = GP\cdot\frac{1}{2}GB = \frac{1}{2}CG\cdot GF = \frac{1}{2}AG\cdot GS = GD\cdot GS$, hence the points E, P, D, S are also concyclic, and $\angle(DP, PG) = \angle(GS, SE)$. The problem may therefore be rephrased as follows:

Given a triangle ABC, let D, E and F respectively denote the midpoints of the sides BC, CA and AB. The circle ABE, respectively, ACF, and the line AD meet again at Q, respectively, S. Prove that $\angle AQF = \angle ASE$ (and $ES = FQ$).



Upon inversion of pole A, the problem reads:

Given a triangle $AE'F'$, let the symmedian from A meet the medians from E' and F' at $K = Q'$ and $L = S'$, respectively. Prove that the angles $AE'L$ and $AF'K$ are congruent.

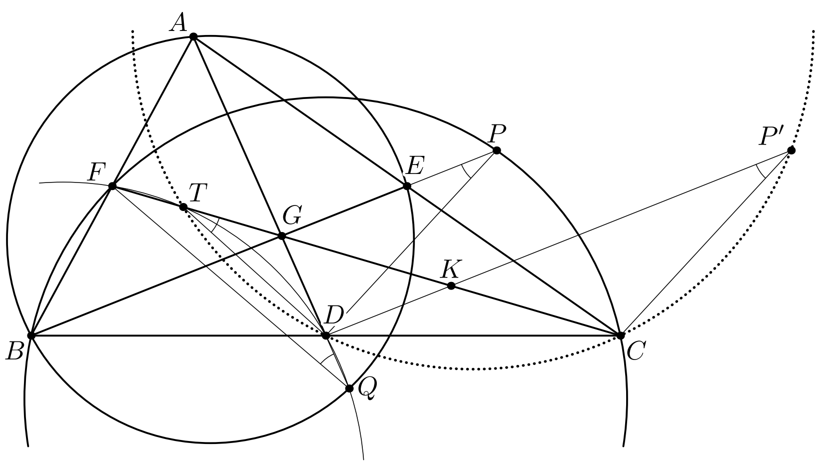


To prove this, denote $E' = X$, $F' = Y$. Let the symmedian from A meet the side XY at V and let the lines XL and YK meet the sides AY and AX at M and N, respectively. Since the points K and L lie on the medians, we have $VM \parallel AX$, $VN \parallel AY$. Hence $AMVN$ is a parallelogram, the symmedian AV of triangle AXY supports the median of triangle AMN, which implies that the triangles AMN and AXY are similar. Hence the points M, N, X, Y are concyclic, and $\angle AXM = \angle AYN$, QED.

Remark 1. We know that the points X, Y, M, N are concyclic. Invert back from A and consider the circles AFQ and AES: the former meets AC again at M' and the latter meets AB again at N'. Then the points E, F, M', N' are concyclic.

Remark 2. The inversion at pole A also allows one to show that $\angle AQF$ is the Brocard angle, thus providing one more solution. In our notation, it is equivalent to the fact that the points Y, K, and Z are collinear, where Z is the Brocard point (so $\angle ZAX = \angle ZYA = \angle ZXY$). This is valid because the lines AV, XK, and YZ are the radical axes of the following circles: (i) passing through X and tangent to AY at A; (ii) passing through Y and tangent to AX at A; and (iii) passing through X and tangent to AY at Y. The point K is the radical center of these three circles.

Solution 3. (Ilya Bogdanov) Again, we will prove that $\angle(DP, PG) = \angle(FQ, QG)$. Mark a point T on the ray GF such that $GF \cdot GT = GQ \cdot GD$; then the points F, Q, D, T are concyclic, and $\angle(FQ, QG) = \angle(TG, TD) = \angle(TC, TD)$.



Shift the point P by the vector $\overrightarrow{BD}$ to obtain point P'. Then $\angle(DP, PG) = \angle(CP', P'D)$, and we need to prove that $\angle(CP', P'D) = \angle(CT, TD)$. This is precisely the condition that the points T, D, C, P' be concyclic.

Denote $GE = x$, $GF = y$. Then $GP \cdot GB = GC \cdot GF$, so $GP = y^2/x$. On the other hand, $GB \cdot GE = GQ \cdot GA = 2GQ \cdot GD = 2GT \cdot GF$, so $GT = x^2/y$. Denote by K the point of intersection of DP' and CT; we need to prove that $TK \cdot KC = DK \cdot KP'$.

Now, $DP' = BP = BG + GP = 2x + y^2/x$, $CT = CG + GT = 2y + x^2/y$, $DK = BG/2 = x$, $CK = CG/2 = y$. Hence the desired equality reads $x(x + y^2/x) = y(y + x^2/y)$ which is obvious.

Remark. The points B, T, E, and C are concyclic, hence the point T is also of the same kind as P and Q.